**The Derivatives Technology Foundation**

**Project Proposal**

" Extensions of the SABR Model for Equity Options "

**Project Description**

The SABR Model for option pricing under stochastic volatility[1] is given by the following stochastic differential equations for the forward stock value f under the riskneutral measure Q:

$$df_t = \alpha_t (f_t)^{\beta} dW_t^1$$
$$d\alpha_t = v \alpha_t dW_t^2$$

where the elasticity coefficient β and the volatility of volatility v are assumed to be known constants. The two Brownian Motions are assumed to be correlated with constant correlation coefficient ρ.

The model seems to derive its popularity from the fact that the (Black-Scholes based) implied volatilities, as a function of strike K for a fixed maturity T and stock price level S, can be obtained in a series expansion which gives a more or less explicit form. Moreover, from these equations one may show that the smile dynamics generated by the model are such that for decreasing (increasing) stock price levels, the smile shifts to the left (the right). This is the behavior usually encountered in the markets, and thus a desirable feature of the model, in contrast to most local volatility models where stock prices and skew curves move in the wrong (opposite) direction of each other. See [2] for a discussion on these and other features of the model.

The equation for the skew in terms of the forward price f equals

$$\sigma_{imp}(f,\alpha,K) = \frac{\alpha}{f^{1-\beta}}\left(1 - \tfrac{1}{2}[1-\beta-\rho\lambda]\log\frac{K}{f} + \tfrac{1}{12}[(1-\beta)^2 + (2-3\rho^2)\lambda^2 \log^2\frac{K}{f} + ..\right)$$

where

$$\lambda = \frac{v}{\alpha} f^{1-\beta}$$

is the ratio between the volatility of volatility v and the local volatility α/f. Note that the accuracy of this expansion depends on the ratio K/f, so in principle the formula is only valid for strikes that are not too far away from the forward price. From this formula we can see that

- The ATM volatility is given by αf^{1- β} so the shape of the 'backbone' of the volatility smile (the relationship between ATM implied vol and stock price level) is controlled by the parameter β and in particular it is flat (in f) when we take β =1, corresponding to lognormal distributions (when conditioned on α).

- The second term models the skew. (i.e. how implied volatilities vary with the strike K). The part -(1/2)(1- β) log (K/f) is skew due to the fact that the local volatility αf^{1- β} is decreasing in f; it disappears when we take β =1. The second part (1/2) ρλ log (K/f) is the part that corresponds to *vanna*, the (negative) correlation between stock price levels and volatility levels. Likewise, the expression (1/12)(2-3$\rho^2$) $\lambda^2$ $\log^2$(K/f ) in the last term can be associated with *volga*, the second derivative of the price with respect to the volatility.

There exists a time-varying version of the model, where the vol of vol and correlation become time-dependent v=v(t), ρ = ρ(t), while the right-hand side of the stochastic differential equation for f is also multiplied by a time-varying function γ(t). All these time-varying functions are still chosen to be deterministic.

Earlier research in TDTF, by Geeske Vlamimg [5], investigated whether American options on equity with cash dividends could be priced in the SABR model when suitable extensions were made to an algorithm developed by Vellekoop & Nieuwenhuis [3] that had been designed for the Heston model. This would allow to test whether the SABR model could improve the fit of the Heston model on market data, especially for small maturities [6].

In that research project it has been shown that the approximating direct formula for implied volatilities does give accurate prices for most situations, but does not give the same European option prices as the numerical methods, when the strike price K is high and when $v^2T$ has a high value. The Vellekoop and Nieuwenhuis method was successfully extended to allow quick and accurate pricing of European as well as American options.

The SABR model seems to have a better fit to market prices of American put options than the Heston model. This is probably caused by the fact that the volatility in the Heston model is assumed to have a normal distribution, while in the SABR model it is assumed to have a lognormal distribution (see more details in [4]). For American options on a single stock, the error between the prices given by the SABR model and the market prices, for a set of only call or only put options with the same time to maturity, is less than 1 cent. This is even true for long maturity times where the error grows, due to higher option prices and larger insecurity. But the error remains within the normal range of the bid-ask spread. The SABR parameters are also more stable over different maturity times than the Heston parameters. In fact, the value of beta does not really seem to influence the quality of the fit at all. Therefore it was chosen to be 1.

In this project, we would like to see if similar results can be established for the following different model (under the riskneutral measure):

$$dS_t = rS_t dt + \alpha_t (S_t)^{\beta} dW_t^1$$
$$d\alpha_t = \kappa(\theta_t - \alpha_t)dt + v\alpha_t dW_t^2$$

which combines feautures of the Heston and SABR model. In particular, we would like to

- design a method which is fast and accurate, to reproduce the European option prices generated by the model that is described by the equations above, based on
  - Monte Carlo Simulations, and/or
  - Two-dimensional trees
- extend such methods for cases where we have options with early-exercise opportunities (ie. American options) or options on underlying assets that pay (cash) dividends at deterministic times in the future.
- Determine whether it is possible to derive (approximate) pricing formulas, or characterization of the backbone for implied volatilities for this model as well.
- Test these methods by using them to
  - fit this model to real market data
  - investigate the stability of the relevant parameters under market conditions once the backbone has been fitted
- If time permits, other models which are hybrids between Heston and SABR models will be investigated as well.

**Starting Literature**


1. Hagan, P.S. & Kumar, D. & Lesniewski, A.S. & Woodward, D.E., Managing Smile Risk, *Wilmott Magazine*, pp. 84-108, 2002.
2. West, G., Calibration of the SABR Model in Illiquid Markets, *Applied Mathematical Finance*, 12(4), pp. 371-385, 2005.
3. Vellekoop, M.H. & Nieuwenhuis, J.W., A tree-based Method to price American Options in the Heston Model. *Preprint*, University of Twente, 2006.

4. Hagan, P.S. & Lesniewski, A.S. & Woodward, D.E., Probability Distribution in the SABR Model of Stochastic Volatility, Working Paper, 2004. Downloadable via www.wilmott.com/attachments/SABR_ProbDistr.zip.
5. Vlaming, G., Pricing Options with the SABR Model, M.Sc. Thesis, University of Utrecht, June 2008.
6. Van der Horst, J., American Option Pricing in the Heston Model: Dealing with Cash Dividends, M.Sc. Thesis, University of Delft, November 2007.

**Proposed Location for Project**
Saen Options

**Proposed Duration for Project**
6 to 9 months

**Supervisor**
Michel Vellekoop, TDTF & University of Twente

**Co-supervisors**
Francois Myburg, TDTF & Saen Options

Feedback may also be sought from the other parties who participate in TDTF.

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